NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Areas and Volumes

Presenter Notes

2016-2017 EDITION

Student Study Session - Presenter Notes

Thank you for agreeing to present at one of NMSI's Saturday Study Sessions. We are grateful you are sharing your time and expertise with our students. Saturday mornings can be a "tough sell" for students, so we encourage you to incorporate strategies and techniques to encourage student movement and engagement. Suggestions for different presentation options are included in this document. If you have any questions about the content or about presentation strategies, please contact Mathematics Director Charla Holzbog at cholzbog@nms.org or AP Calculus Content Specialist Karen Miksch at kmiksch@nms.org.

The material provided contains many released AP multiple choice and free response questions as well as some AP-like questions that we have created. The goal for the session is to let the students experience a variety of both types of questions to gain insight on how the topic will be presented on the AP exam. It is also beneficial for the students to hear a voice other than their teacher in order to help clarify their understanding of the concepts.

Suggestions for presenting:
The vast majority of the study sessions are on Saturday and students and teachers are coming to be WOWed! We want activities to engage the students as well as prepare them for the AP Exam. The following presenter notes include pacing suggestions (you only have 50 minutes!), solutions, and recommended engagement strategies.

Suggestions on how to prepare:

- The notes/summaries on the last page(s) are for reference. We want the students' time during the session to be focused on the questions as much as possible and not taking or reading the notes. As the questions are presented during the session, you may wish to refer the students back to those pages as needed. It is not our intent for the sessions to begin with a lecture over these pages.
- As you prepare, work through the questions in the packet noting the level of difficulty and topic or skill required for the questions.
- Design a plan for what questions you would like to cover with the group depending on their level of expertise. Some groups will be ready for the tougher questions while other groups will need more guidance and practice on the easier ones. Create an easy, medium, and hard listing of the questions prior to the session. This will allow you to adjust on the fly as you get to know the groups. In most instances, there will **not** be enough time to cover all the questions in the packet. Use your judgement on the amount of questions to cover based on the students' interactions. Remember to include both multiple choice and free response type questions. Discussions on test taking strategies and scoring of the free response questions are always great to include during the day.
- The concepts should have been previously taught; however, be prepared to "teach" the topic if you find out the students have not covered the concept prior in class. In sessions where multiple schools come together, you might have a mixture of students with and without prior knowledge on the topic. You will have to use your best judgement in this situation.
- Consider working through some free response questions before the multiple choice questions, or flipping back and forth between the two types of questions. Sometimes, if free response questions are saved for the last part of the session, it is possible students only get practice with one or two of them and most students need additional practice with free response questions.

Areas and Volumes Presenter Notes

This session includes a reference sheet at the back of the packet. We suggest that the presenter not spend time going over the reference sheet, but point it out to students so that they may refer to it if needed.

We suggest that students will work in small groups of 3 or 4 (depending on the size of the class)- arrange the desks prior to the start of the session.

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use our presenter notes and your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted. Notice in the solutions guide the questions are categorized as 3, 4, or 5 indicating a typical question of the difficulty level (DL) for a student earning these qualifying scores on the AP exam.

I. 10 Minute Group Activity

- Arrange the students randomly into groups of 3 or 4. Ask students to work together on free response question 12 while you actively walk around to monitor their progress and assist with any questions. Use this time to gauge the level of knowledge of the group.
- After a few minutes, review the answers with students and clarify any misunderstandings. Display the scoring rubric for the students and discuss what is necessary to earn points on the AP exam.

II. 20 minutes Multiple Choice Practice-*your choice of questions depending on student struggles during the first ten minutes—some suggested questions are listed below.*

- Choose a couple multiple choice questions to model for the students, and then choose several to assign to the groups as time permits. Selections should be based on the knowledge level of the group. Use a think-pair-share strategy for the assigned questions. Suggestion: use 3 and 7 as models, then assign 1, 2, 5 and 6 (include 4 and 8 for stronger groups).

III. 20 minutes additional Free Response Practice

- Assign questions 9 and 10 to the groups. Allow time for some discussion and collaboration within the groups before reviewing the answers with the students.
- Assign remaining free response questions (11 and 13) as time permits.

Multiple Choice

1. C (1993 AB30) DL: 3
   Each slice is a disk whose volume is given by $\pi r^2 \Delta x$ where $r = \sqrt{x}$.
   $$V = \pi \int_0^3 \left(\sqrt{x}\right)^2 \, dx = \pi \int_0^3 x \, dx = \frac{\pi}{2} x^2 \Big|_0^3 = \frac{9\pi}{2}$$

2. A (2012 AB10) DL: 4
   $$\int_{0`}^{2} e^{\frac{x}{2}} \, dx = 2\int_{0`}^{2} \frac{1}{2} e^{\frac{x}{2}} \, dx = 2e^{\frac{x}{2}} \Big|_0^2 = 2e - 2$$

3. B (2008 AB83/BC83) DL: 3
   Graph the curves to see that they intersect at $x = 1,\ 2,$ and 5.
   Let $f(x) = x^3 - 8x^2 + 18x - 5$ and $g(x) = x + 5$.
   Area = $\int_1^2 (f(x) - g(x)) \, dx + \int_2^5 (g(x) - f(x)) dx = 11.833$
   Or Area = $\int_1^5 \left| f(x) - g(x) \right| dx = 11.833$

4. C (1969 AB13) DL: 5
   $$\int_{-\frac{\pi}{2}}^{k} \cos x \, dx = 3\int_{k}^{\frac{\pi}{2}} \cos x \, dx$$
   $$\sin k - \sin\left(-\frac{\pi}{2}\right) = 3(\sin\frac{\pi}{2} - \sin k)$$
   $$\sin k + 1 = 3 - 3\sin k$$
   $$4\sin k = 2$$
   $$\sin k = \frac{1}{2}$$
   $$k = \frac{\pi}{6}$$

5. C (AP-like) DL: 3
   $y = \sqrt{x} \quad \Rightarrow \quad x = y^2$
   $y = x^2 \ \Rightarrow \ x = \sqrt{y}$ since $x > 0$
   $y^2 = \sqrt{y}$ at $y = 0$ and $y = 1$
   Volume $= \int_0^1 \left(\sqrt{y} - y^2\right)^2 dy$

6. B (from 2007 Free Response AB1 c) DL: 4

$\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$

Volume $= \dfrac{\pi}{2}\displaystyle\int_{-3}^{3} r^2 dx$ where $r = \dfrac{1}{2}\left(\dfrac{20}{1+x^2} - 2\right)$

Volume $= \dfrac{\pi}{2}\displaystyle\int_{-3}^{3}\left(\dfrac{1}{2}\left(\dfrac{20}{1+x^2} - 2\right)\right)^2 dx$

Or Volume $= \dfrac{\pi}{8}\displaystyle\int_{-3}^{3}\left(\dfrac{20}{1+x^2} - 2\right)^2 dx \quad = 174.268$

7. A (AP-like) DL: 5

For $x = 0$ to $x = 3$, the volume is calculated using the disc method.

Volume $= \pi\displaystyle\int_0^3 r^2 dx$ where $r = 2\sqrt{x}$

so Volume $= \pi\displaystyle\int_0^3 \left(2\sqrt{x}\right)^2 dx = 4\pi\int_0^3 x\, dx$

For $x = 3$ to $x = 9$, the volume is calculated using the washer method.

Volume $= \pi\displaystyle\int_3^9 \left(R^2 - r^2\right) dx$ where $R = 2\sqrt{x}$ and $r = x - 3$

so Volume $= \pi\displaystyle\int_3^9 \left[\left(2\sqrt{x}\right)^2 - \left(x-3\right)^2\right] dx = \pi\int_3^9\left[4x - \left(x-3\right)^2\right] dx$

Therefore, the total volume is given by $4\pi\displaystyle\int_0^3 x\, dx + \pi\int_3^9\left[4x - \left(x-3\right)^2\right] dx$

8. A (AP-like) DL: 5

$\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k^{*}\right)\Delta x_k$ where $\Delta x_k = \dfrac{b-a}{n}$ and x_k^{*} is a value in the kth subinterval.

The area can be calculated by the definite integral $\displaystyle\int_1^3 \ln x dx$.

This definite integral can be written as $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{2k}{n}\right)\frac{2}{n}$ since $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$ and $1 + \dfrac{2k}{n}$ is a value in the kth subinterval.

Free Response
9. 2014 AB2

(a) $f(x) = 4 \Rightarrow x = 0, 2.3$

volume
$$= \pi \int_0^{2.3} \left[(4+2)^2 - (f(x)+2)^2 \right] dx$$
$= 98.868$ (or 98.867)

4: 2: integrand
1: limits
1: answer

(b) Volume $= \int_0^{2.3} \frac{1}{2}(4 - f(x))^2 dx$
$= 3.574 (or 3.573)$

3: 2: integrand
1: answer

(c) $\int_0^k (4 - f(k))dx = \int_k^{2.3} (4 - f(x))\, dx$

2: 1: area of one region
1: equation

10. 2013 AB5

(a) $\text{Area} = \int_0^2 [g(x) - f(x)]dx$

$$= \int_0^2 \left[4\cos\left(\frac{\pi}{4}x\right) - (2x^2 - 6x + 4) \right] dx$$

$$= \left[4 \bullet \frac{4}{\pi} \sin\left(\frac{\pi}{4}x\right) - \left(\frac{2x^3}{3} - 3x^2 + 4x\right) \right]_0^2$$

$$= \frac{16}{\pi} - (\frac{16}{3} - 12 + 8) = \frac{16}{\pi} - \frac{4}{3}$$

4: $\begin{cases} \text{1: integrand} \\ \text{2: antiderivative} \\ \text{1: answer} \end{cases}$

(b) Volume $= \pi \int_0^2 \left[(4 - f(x))^2 - (4 - g(x))^2 \right] dx$

$$= \pi \int_0^2 \left[\left(4 - \left(2x^2 - 6x + 4\right)\right)^2 - \left(4 - 4\cos\left(\frac{\pi}{4}x\right)\right)^2 \right] dx$$

3: $\begin{cases} \text{2: integrand} \\ \text{1: limits and constant} \end{cases}$

(c) Volume $= \int_0^2 [g(x) - f(x)]^2 \, dx$

$$= \int_0^2 \left[4\cos\left(\frac{\pi}{4}x\right) - \left(2x^2 - 6x + 4\right) \right]^2 dx$$

2: $\begin{cases} \text{1: integrand} \\ \text{1: limits and constant} \end{cases}$

11. 2011B AB3ab

(a) $\text{Area} = \int_0^4 \sqrt{x} \, dx + \frac{1}{2} \cdot 2 \cdot 2 = \frac{2}{3} x^{3/2} \Big|_{x=0}^{x=4} + 2 = \frac{22}{3}$

3 $\begin{cases} \text{1: integral} \\ \text{1: antiderivative} \\ \text{1: answer} \end{cases}$

(b) $y = \sqrt{x} \Rightarrow x = y^2$

$y = 6 - x \Rightarrow x = 6 - y$

$\text{Width} = (6 - y) - y^2$

$\text{Volume} = \int_0^2 2y\left(6 - y - y^2\right) dy$

3 $\begin{cases} \text{2: integrand} \\ \text{1: answer} \end{cases}$

12. 2009 AB4

(a) $\text{Area} = \int_0^2 \left[2x - x^2\right] dx$ $= x^2 - \frac{1}{3}x^3 \Big\vert_{x=0}^{x=2}$ $= \frac{4}{3}$	3: 1: integrand 1: antiderivative 1: answer
(b) $\text{Volume } = \int_0^2 \sin\left(\frac{\pi}{2}x\right) dx$ $= \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big\vert_{x=0}^{x=2}$ $= \frac{4}{\pi}$	3: 1: integrand 1: antiderivative 1: answer
(c) $\text{Volume } = \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)^2 dx$	3: 2: integrand 1: limits

13. 2008 AB1/BC1 modified part c

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ $\text{Area} = \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	3 1: limits 1: integrand 1: answer
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	2 1: limits 1: integrand
(c) $\text{Volume} = \frac{\sqrt{3}}{4}\int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 4.320 \text{ or } 4.321$	2 1: integrand 1: answer
(d) $\text{Volume} = \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369 \text{ or } 8.370$	2 1: integrand 1: answer

